

1. Evaluate the following integral.

$$\int_9^{4y} \frac{6y}{x} dx = \left[ 6y \ln|x| \right]_{x=9}^{x=4y} = 6y (\ln(4y) - \ln 9) = 6y \ln\left(\frac{4y}{9}\right)$$

(assume  $y > 0$   
otherwise  
integral diverges)

2. Evaluate the following integral.

$$\int_{4x}^{x^4} \frac{-10y}{x} dy = \left[ \frac{-5y^2}{x} \right]_{y=4x}^{y=x^4} = -5x^7 - \frac{-5(4x)^2}{x} = 80x - 5x^7$$

3. Evaluate the following integral.

$$\int_y^{\frac{\pi}{11}} \sin^3 3x \cos y dx = \cos y \int_y^{\frac{\pi}{11}} (1 - \cos^2 3x) \sin 3x dx$$

$u = \cos 3x$   
 $du = -3 \sin(3x) dx$

$$\cos y \left[ u - \frac{u^3}{3} \right]_{x=y}^{x=\frac{\pi}{11}} = \frac{1}{3} \cos y \left( \cos 3y + \frac{\cos^3(3y)}{3} - \cos\left(\frac{3\pi}{11}\right) + \frac{\cos^3\left(\frac{3\pi}{11}\right)}{3} \right)$$

**REVISED**  
8:14 pm, 4/24/06

4. Evaluate the following iterated integral.

$$\int_2^6 \int_1^6 (4x+y) dy dx$$

$$\int_2^6 \left[ 4xy + \frac{y^2}{2} \right]_{y=1}^{y=6} dx = \int_2^6 \left( 24x + \frac{6^2}{2} - \left( 4x + \frac{1}{2} \right) \right) dx$$

$$= \int_2^6 \left( 20x + \frac{35}{2} \right) dx = \left[ 10x^2 + \frac{35x}{2} \right]_2^6 = 360 + 105 - 40 - 35 = 390$$

5. Evaluate the following iterated integral.

$$\int_6^7 \int_1^{\sqrt{x}} 2ye^{-x} dy dx = \int_6^7 \left[ y^2 e^{-x} \right]_{y=1}^{y=\sqrt{x}} dx = \int_6^7 x e^{-x} - e^{-x} dx = \left[ -x e^{-x} \right]_6^7 = \frac{6}{e^6} - \frac{7}{e^7}$$

6. Evaluate the following iterated integral.

$$\int_3^5 \int_y^{-2y} (-2+6x^2+6y^2) dx dy = \int_3^5 \left[ -2x + 2x^3 + 6xy^2 \right]_{x=y}^{x=-2y} dy = \int_3^5 (-36y^3 + 6y) dy = \left[ -9y^4 + 3y^2 \right]_3^5 = -9 \cdot 625 + 75 + 9 \cdot 81 - 27 = -5625 + 75 + 729 - 27 = -4848$$

7. Evaluate the following iterated integral.

$$\int_0^{\frac{\pi}{9}} \int_0^{11 \cos \theta} r dr d\theta = \int_0^{\frac{\pi}{9}} \left[ \frac{r^2}{2} \right]_0^{11 \cos \theta} d\theta = \int_0^{\frac{\pi}{9}} \frac{121 \cos^2 \theta}{2} d\theta = \frac{121}{4} \left[ \theta + \sin \theta \cos \theta \right]_0^{\frac{\pi}{9}} = \frac{121 \pi}{36} + \frac{121}{8} \sin\left(\frac{2\pi}{9}\right)$$

8. Evaluate the following improper integral.

$$\int_2^{\infty} \int_0^{\frac{7}{x}} y dy dx = \int_2^{\infty} \left[ \frac{y^2}{2} \right]_0^{\frac{7}{x}} dx = \int_2^{\infty} \frac{49}{2x^2} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{49}{2x^2} dx = \lim_{b \rightarrow \infty} \left[ -\frac{49}{2x} \right]_2^b = \frac{49}{4}$$

9. Use an iterated integral to find the area of the region bounded by

$\sqrt{x} + \sqrt{y} = 2, x=0, y=0.$

$$\int_{x=0}^4 \int_{y=0}^{(2-\sqrt{x})^2} dy dx = \int_0^4 (2-\sqrt{x})^2 dx$$

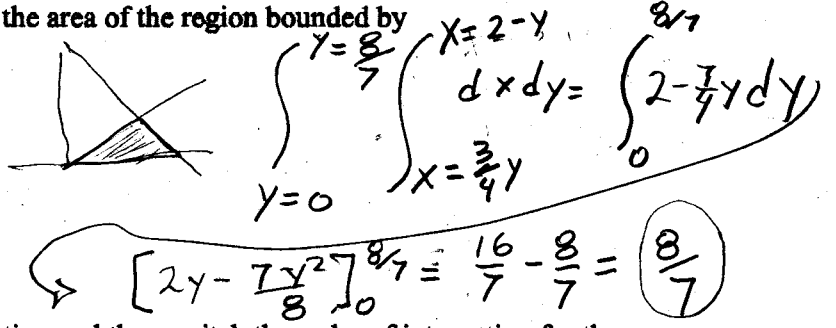
$$= \int_0^4 4 - 2\sqrt{x} + x dx = \left[ 4x - \frac{4x^{3/2}}{3} + \frac{x^2}{2} \right]_0^4 = 16 - \frac{32}{3} + 8 = \frac{40}{3}$$

10. Use an iterated integral to find the area of the region bounded by

intersect

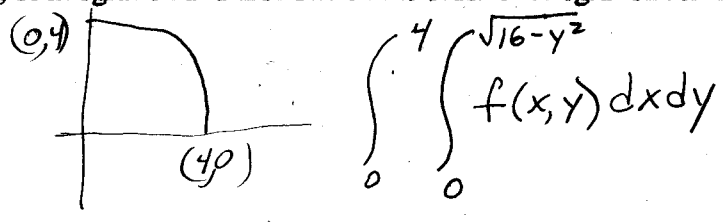
$4x - 3y = 0, x + y = 2, y = 0.$

$4x - 3(2-x) = 0$   
 $7x - 6 = 0$   
 $x = \frac{6}{7}, y = \frac{8}{7}$



11. Sketch the region, R, of integration and then switch the order of integration for the following integral.

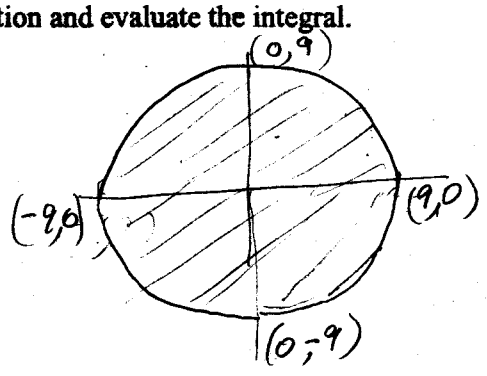
$$\int_0^4 \int_0^{\sqrt{16-x^2}} f(x,y) dy dx$$



$$\int_0^4 \int_0^{\sqrt{16-y^2}} f(x,y) dx dy$$

12. Sketch the region, R, whose area is given by the iterated integral below. Then switch the order of integration and evaluate the integral.

$$\int_{-9}^9 \int_{-\sqrt{81-x^2}}^{\sqrt{81-x^2}} dy dx$$



$$\int_{-9}^9 \int_{-\sqrt{81-y^2}}^{\sqrt{81-y^2}} dx dy$$

$$\int_{-9}^9 2\sqrt{81-y^2} dy = \left[ y\sqrt{81-y^2} + 81 \arcsin \frac{y}{9} \right]_{-9}^9 = \frac{81\pi}{2} - -\frac{81\pi}{2} = 81\pi$$

(integration formula a 37 pg A29)

13. Evaluate the iterated integral below. Note that it is necessary to switch the order of integration.

$$\int_0^5 \int_x^5 e^{-8y^2} dy dx \quad \begin{array}{l} x=0 \\ y=5 \\ y=x \\ (0,0) \end{array} \quad \int_0^5 \int_0^y e^{-8y^2} dx dy = \int_0^5 y e^{-8y^2} dy$$

$$u = -8y^2 \\ du = -16y dy \quad -\frac{1}{16} \int_{y=0}^{y=5} e^u du = -\frac{1}{16} [e^{-8y^2}]_0^5 = \frac{1}{16} - e^{-200}$$

14. Sketch the region  $R$  and evaluate the iterated integral.

$$I = \int_0^6 \int_{x/2}^3 (x+y) dy dx \quad \begin{array}{l} (0,3) \\ y=3 \\ x=0 \\ (6,3) \\ y=x/2 \\ (0,0) \end{array} \quad \int_0^6 \left[ xy + \frac{y^2}{2} \right]_{y=x/2}^{y=3} dx$$

$$\int_0^6 \left( 3x + \frac{9}{2} - \frac{x^2}{2} - \frac{x^2}{8} \right) dx = \left[ \frac{5x^2}{2} + \frac{9x}{2} - \frac{x^3}{6} \right]_0^6 = \frac{5 \cdot 36}{2} + \frac{3 \cdot 18}{2} - \frac{216}{6} = 90 + 27 - 36 = 81$$

15. Set up an integral for both orders of integration, and use the more convenient order to evaluate the integral below over the region  $R$ .

$$\iint_R \frac{y}{x^2+y^2} dA \quad \begin{array}{l} y=8x \\ x=5 \\ y=5x \end{array} \quad \int_0^5 \int_{y/8}^{y/5} \frac{y}{x^2+y^2} dx dy \quad \text{better is } \int_0^5 \int_{y=5x}^{y=8x} \frac{y}{x^2+y^2} dy dx$$

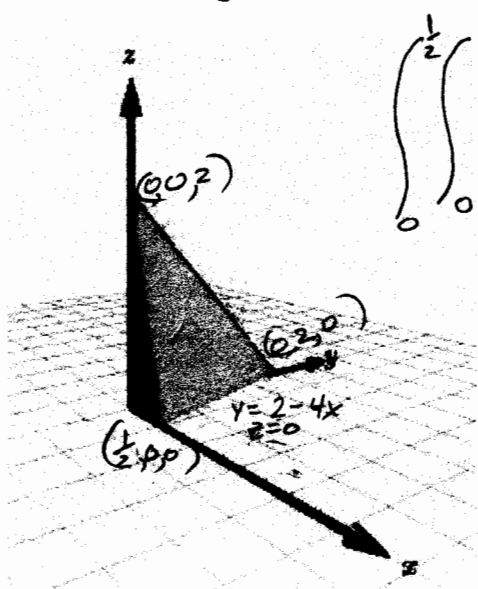
$R$ : triangle bounded by  $y=5x$ ,  $y=8x$ , and  $x=5$

$$\text{let } u = x^2 + y^2 \\ du = 2y dy \quad \int_0^5 \int_{u=26x^2}^{u=65x^2} \frac{du}{2u} = \int_0^5 \frac{1}{2} \left[ \ln u \right]_{26x^2}^{65x^2} dx = \int_0^5 \frac{1}{2} \ln \frac{65}{26} dx = \frac{5}{2} \ln \frac{5}{2}$$

- 16.

OMIT

17. Use a double integral to find the volume of the indicated solid.



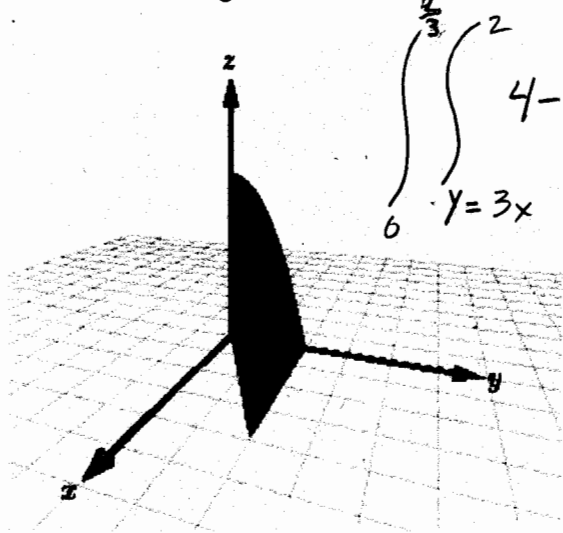
$$\int_0^{\frac{1}{2}} \int_0^{2-4x} z \, dy \, dx = \int_0^{\frac{1}{2}} \int_0^{2-4x} (2-4x-y) \, dy \, dx = \int_0^{\frac{1}{2}} \left[ 2y - 4xy - \frac{y^2}{2} \right]_{y=0}^{y=2-4x} dx$$

$$\int_0^{\frac{1}{2}} \left( 2(2-4x) + 4x(2-4x) - \frac{(2-4x)^2}{2} \right) dx$$

$$\int_0^{\frac{1}{2}} (-24x^2 + 8x + 2) dx = \left[ -8x^3 + 4x^2 + 2x \right]_0^{\frac{1}{2}} = 1$$

$4x + y + z = 2, x > 0, y > 0, z > 0$

18. Use a double integral to find the volume of the indicated solid.



$$\int_0^{\frac{2}{3}} \int_{3x}^2 (4-y^2) \, dy \, dx = \int_0^{\frac{2}{3}} \left[ 4y - \frac{y^3}{3} \right]_{y=3x}^{y=2} dx$$

$$= \int_0^{\frac{2}{3}} \left( 8 - \frac{8}{3} + 12x - 9x^3 \right) dx$$

$$\left[ \frac{16x}{3} - 6x^2 + \frac{9x^4}{4} \right]_0^{\frac{2}{3}}$$

$z = 4 - y^2, z > 0, x > 0, 3x < y < 2$

$$= \frac{32}{9} - \frac{24}{9} + \frac{4}{9} = \frac{12}{9} = \frac{4}{3}$$

**REVISED**  
8:15 pm, 4/24/06

19. Set up a double integral to find the volume of the solid bounded by the graphs of the equations given below.

$$z = xy, \quad z > 0, \quad x > 0, \quad 7x < y < 4$$

$$\int_0^{\frac{4}{7}} \int_{7x}^4 xy \, dy \, dx = \int_0^{\frac{4}{7}} \left[ \frac{xy^2}{2} \right]_{y=7x}^{y=4} dx$$

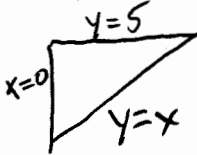
$$\int_0^{\frac{4}{7}} 8x - \frac{49x^3}{2} dx = \left[ 4x^2 - \frac{49x^4}{8} \right]_0^{\frac{4}{7}} = 4\left(\frac{4}{7}\right)^2 - \frac{49}{8}\left(\frac{4}{7}\right)^4 = \frac{32}{49}$$

20. Set up a double integral to find the volume of the solid bounded by the graphs of the equations given below.

$$z = \frac{1}{25+y^2}, \quad x=0, \quad x=2, \quad y \geq 0$$

$$\int_0^2 \int_0^{\infty} \frac{1}{25+y^2} dy \, dx$$

21. Evaluate the iterated integral below. Note that it is necessary to switch the order of integration.

$$\int_0^5 \int_x^5 e^{-0.29y^2} dy \, dx$$


$$\int_0^5 \int_0^y e^{-0.29y^2} dx \, dy = \int_0^5 ye^{-0.29y^2} dy$$

$$u = -0.29y^2$$

$$du = -0.58y \, dy$$

$$-\frac{1}{0.58} \int_{y=0}^{y=5} e^u du = -\frac{1}{0.58} \left[ e^{-0.29y^2} \right]_{y=0}^{y=5} = \frac{1}{0.58} (1 - e^{-8.41})$$

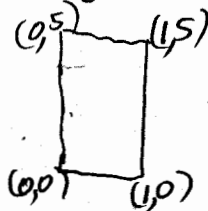
22. Find the average value of  $f(x,y)$  over the region  $R$  where:

$$\text{Average value} = \frac{1}{A} \iint_R f(x,y) dA$$

$$f(x,y) = xy$$

$$A = 5$$

$R$ : rectangle with vertices  $(0,0)$ ,  $(1,0)$ ,  $(1,5)$ ,  $(0,5)$



$$\frac{1}{5} \int_0^1 \int_0^5 xy \, dy \, dx = \int_0^1 \left[ \frac{xy^2}{2} \right]_0^5 dx = \frac{1}{5} \int_0^1 \frac{25x}{2} dx$$

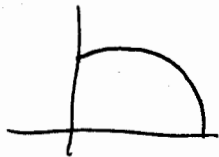
$$\frac{1}{5} \int_0^1 \frac{25x}{2} dx = \left[ \frac{25x^2}{5 \cdot 4} \right]_0^1 = \frac{5}{4}$$

**REVISED**

8:16 pm, 4/24/06

23. Identify the region of integration for the following integral.

$$\int_0^{\pi/2} \int_0^{1+\sin\theta} r \theta dr d\theta \quad \text{First quadrant of the Cardioid}$$



24. Evaluate the double integral below.

$$\int_0^{2\pi} \int_0^6 5r^5 \sin\theta dr d\theta = \int_0^{2\pi} \left[ \frac{5r^6}{6} \right]_0^6 \sin\theta d\theta = 5 \cdot 6^5 \left[ -\cos\theta \right]_0^{2\pi} = 0$$

25. Evaluate the double integral below.

$$\int_0^3 \int_0^{\pi/4} r^3 \sin\theta \cos\theta dr d\theta = \left[ \frac{r^4}{4} \right]_0^3 \int_0^{\pi/4} \sin\theta \cos\theta d\theta \quad \begin{array}{l} \text{let } u = \sin\theta \\ du = \cos\theta d\theta \end{array}$$

$$\left[ \frac{r^4}{4} \right]_0^3 \left[ \frac{u^2}{2} \right]_{\theta=0}^{\theta=\pi/4} = \frac{81}{4} \sin^2 \frac{\pi}{4} = 1.9$$

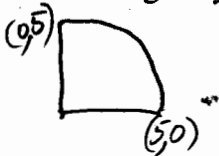
26. Evaluate the double integral below

$$\int_0^{\pi/2} \int_0^{5+3\sin\theta} \theta r dr d\theta = \int_0^{\pi/2} \left[ \frac{\theta r^2}{2} \right]_0^{5+3\sin\theta} d\theta$$

$$\frac{1}{2} \int_0^{\pi/2} \theta (5+3\sin\theta)^2 d\theta = 34.32$$

27. Evaluate the following iterated integral by converting to polar coordinates.

$$\int_0^5 \int_0^{\sqrt{25-x^2}} y \, dy \, dx$$



$$\int_0^5 \int_0^{\pi/2} r^2 \sin \theta \, d\theta \, dr = \int_0^5 r^2 [-\cos \theta]_0^{\pi/2} \, dr$$

$$\int_0^5 r^2 \, dr = \left[ \frac{r^3}{3} \right]_0^5 = \frac{125}{3}$$

28.

OMIT

29.

OMIT

30. Use a double integral in polar coordinates to find the volume of the solid in the first octant bounded by the graphs of the equations given below.

$$z = x^4 y, \quad x^2 + y^2 = 16$$

$$\int_0^4 \int_0^{\pi/2} (r \cos \theta)^4 (r \sin \theta) \, r \, d\theta \, dr$$

$$\int_0^4 r^6 \int_0^{\pi/2} \cos^4 \theta \sin \theta \, d\theta \, dr = \left[ -\frac{\cos^5 \theta}{5} \right]_0^{\pi/2} \int_0^4 r^6 \, dr = \frac{1}{5} \left[ \frac{r^7}{7} \right]_0^4 = \frac{4^7}{5 \cdot 7}$$

**REVISED**

8:16 pm, 4/24/06

31. Use a double integral in polar coordinates to find the volume of the solid inside the hemisphere

$$z = \sqrt{81 - x^2 - y^2}$$

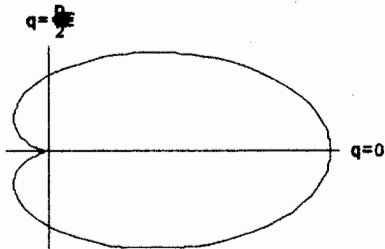
but outside the cylinder

$$\int_0^{2\pi} \int_7^9 \sqrt{81 - r^2} r dr d\theta$$

$$\begin{aligned} \text{let } u &= 81 - r^2 \\ du &= -2r dr \end{aligned}$$

$$2\pi \int_{r=7}^{r=9} \frac{\sqrt{u}}{-2} du = 2\pi \left[ -\frac{u^{3/2}}{3} \right]_{r=7}^{r=9} = \frac{2\pi}{3} 32 \approx 379.13$$

32. Use a double integral to find the area enclosed by the graph of  $r = 5 + 5\cos\theta$ .



$$\int_0^{2\pi} \int_0^{5+5\cos\theta} r dr d\theta = \int_0^{2\pi} \left[ \frac{r^2}{2} \right]_0^{5+5\cos\theta} d\theta$$

$$\int_0^{2\pi} \frac{25}{2} (1 + \cos\theta)^2 d\theta \approx 117.8$$

33.

OMIT

34. Find the area of the surface given by  $z = f(x, y)$  over the region  $R$ .

$$f(x, y) = -6 - 4x + 2y$$



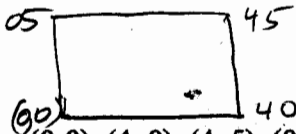
$$f_x = -4 \quad f_y = 2$$

$R$ : square with vertices  $(0, 0)$ ,  $(4, 0)$ ,  $(4, 4)$ ,  $(0, 4)$

$$\int_0^4 \int_0^4 \sqrt{1 + f_x^2 + f_y^2} dx dy = \int_0^4 \int_0^4 \sqrt{1 + 16 + 4} = \sqrt{21} \cdot 16 \approx 73.32$$

35. Find the area of the surface given by  $z = f(x,y)$  over the region  $R$ .

$$f(x,y) = 10 + 2x - 4y$$



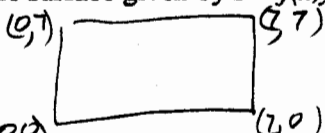
$$f_x = 2 \quad f_y = -4$$

$R$ : rectangle with vertices  $(0,0), (4,0), (4,5), (0,5)$

$$\int_0^4 \int_0^5 \sqrt{1 + f_x^2 + f_y^2} \, dy \, dx = \int_0^4 \int_0^5 \sqrt{1 + 16 + 4} \, dy \, dx = \sqrt{21} \cdot 20 \approx 91.65$$

36. Find the area of the surface given by  $z = f(x,y)$  over the region  $R$ .

$$f(x,y) = -1 - x^2$$



$$f_x = -2x \quad f_y = 0$$

$R$ : rectangle with vertices  $(0,0), (7,0), (7,7), (0,7)$

$$\int_0^7 \int_0^7 \sqrt{1 + 4x^2} \, dx \, dy = 14 \left[ \frac{1}{2} \left( x \sqrt{x^2 + \frac{1}{4}} + \frac{1}{4} \ln \left| x + \sqrt{x^2 + \frac{1}{4}} \right| \right) \right]_0^7 \approx 349.7$$

37. Find the area of the surface given by  $z = f(x,y)$  over the region  $R$ .

$$f(x,y) = xy \quad f_x = y \quad f_y = x$$

$$R: \{(x,y) : x^2 + y^2 \leq 121\}$$

$$\iint_R \sqrt{1 + x^2 + y^2} \, dA = \int_0^{2\pi} \int_0^{11} \sqrt{1 + r^2} \, r \, dr \, d\theta = \pi \int_0^{11} \sqrt{u} \, du = \pi \left[ \frac{2u^{3/2}}{3} \right]_0^{122} = 897.689\pi$$

38. Find the area of the surface of the portion of the plane

$$z = 5 - 2x - 8y$$

$$f_x = -2 \quad f_y = -8$$

in the first octant.

$$R: \{(x,y) \mid 2x + 8y \leq 5 \text{ and } x \geq 0 \text{ and } y \geq 0\}$$

$$\int_0^{\frac{5}{8}} \int_{x=0}^{x=\frac{5-8y}{2}} \sqrt{1 + 4 + 64} \, dx \, dy = \frac{\sqrt{71}}{2} \int_0^{\frac{5}{8}} (5 - 8y) \, dy = \left[ 5y - 4y^2 \right]_0^{\frac{5}{8}} \frac{\sqrt{71}}{2}$$

$$= \left( \frac{25}{8} - \frac{100}{64} \right) \frac{\sqrt{71}}{2} = \frac{50}{64} \sqrt{71} = \frac{25}{32} \sqrt{71}$$

$$\approx 6.58$$

39. Set up a double integral that gives the area of the surface of the graph of  $f$  over the region  $R$ .

$$f(x, y) = x^4 - 3xy + 7y^3 \quad f_x = 4x^3 - 3y \quad f_y = -3x + 21y^2$$

$$R = \{(x, y) : -5 \leq x \leq 5, -2 \leq y \leq 2\}$$

$$\int_{-5}^5 \int_{-2}^2 \sqrt{1 + (4x^3 - 3y)^2 + (-3x + 21y^2)^2} dy dx$$

40. Set up a double integral that gives the area of the surface of the graph of  $f$  over the region  $R$ .

$$f(x, y) = e^{7xy} \quad f_x = 7ye^{7xy} \quad f_y = 7xe^{7xy}$$

$$R = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 4\}$$

$$\int_0^2 \int_0^4 \sqrt{1 + 49y^2 e^{14xy} + 49x^2 e^{14xy}} dy dx$$

41. Evaluate the following iterated integral.

$$\begin{aligned} \int_0^6 \int_0^4 \int_0^5 (5x + 3y + z) dx dy dz &= \int_0^6 \int_0^4 \left[ \frac{5x^2}{2} + 3xy + xz \right]_{x=0}^{x=5} dy dz = \int_0^6 \int_0^4 \left( \frac{125}{2} + 15y + 5z \right) dy dz \\ &= \int_0^6 \left[ \frac{125y}{2} + \frac{15y^2}{2} + 5yz \right]_{y=0}^{y=4} dz = \int_0^6 (250 + 120 + 20z) dz = \left[ 370z + 10z^2 \right]_0^6 = 2580 \end{aligned}$$

42. Evaluate the following iterated integral.

$$\begin{aligned} \int_1^8 \int_0^1 \int_0^x 2ze^{-x^2} dy dx dz &= \int_1^8 \int_0^1 \left[ 2zye^{-x^2} \right]_{y=0}^{y=x} dx dz = \int_1^8 \int_0^1 2xe^{-x^2} dx dz \\ u = -x^2 \quad du = -2x dx & \int_1^8 \int_0^1 -e^u du dz = \int_1^8 \left[ -e^u \right]_0^1 dz = \int_1^8 z(1 - e^{-1}) dz \\ &= \left[ \frac{z^2}{2} (1 - e^{-1}) \right]_1^8 = \frac{63(1 - \frac{1}{e})}{2} \end{aligned}$$

43. Set up a triple integral for the volume of the solid bounded by the coordinate planes and the plane given below.

$$z = 6 - 2x - 4y$$

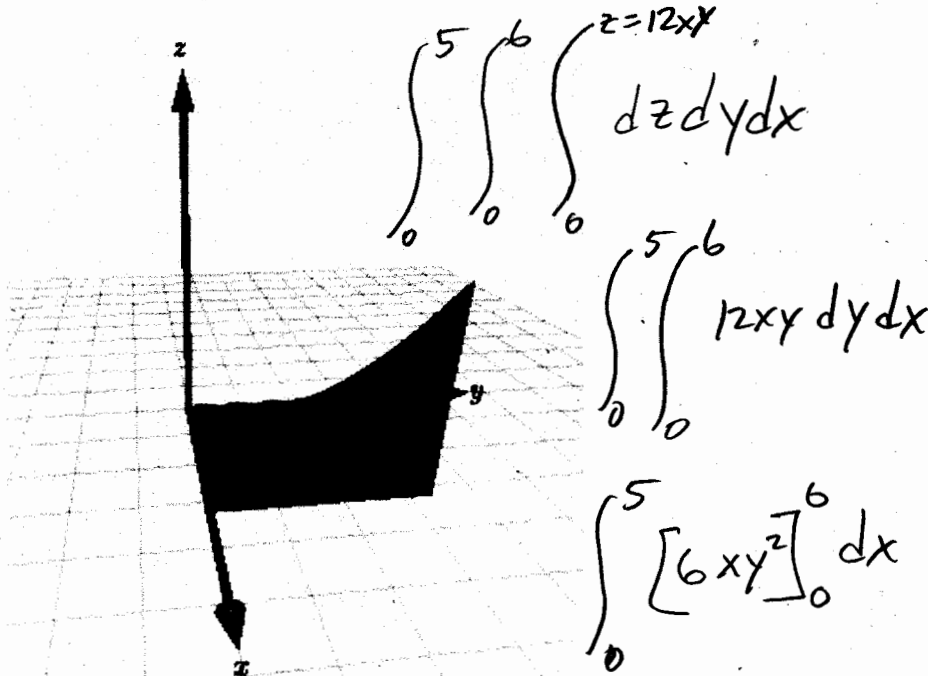
$$\int_0^{\frac{3}{2}} \int_0^{3-2y} \int_0^{6-2x-4y} dz dx dy$$

$y=0$   $x=0$   $z=0$

**REVISED**

8:16 pm, 4/24/06

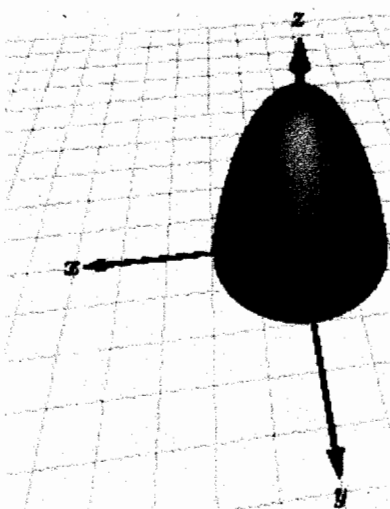
44. Use a triple integral for the volume of the solid shown below.



$$z = 12xy, 0 \leq x \leq 5, 0 \leq y \leq 6$$

$$\left[ 6 \frac{x^2}{2} \right]_0^6 \Big|_0^5 = \frac{6 \cdot 125}{2} = 2700$$

45. Use a triple integral for the volume of the solid shown below.



$$z = 64 - x^2 - y^2, z \geq 0$$

$$\int_{-8}^8 \int_{-\sqrt{64-x^2}}^{\sqrt{64-x^2}} \int_0^{64-x^2-y^2} dz dy dx$$

$$= \int_0^{2\pi} \int_0^8 (64-r^2) r dr d\theta$$

$$2\pi \left[ 32r^2 - \frac{r^3}{3} \right]_0^8 = 2\pi \left( 32 \cdot 64 - \frac{512}{3} \right)$$

$$\approx 11795$$

46. Sketch the solid whose volume is given by the iterated integral given below and re-write the integral using the indicated order of integration.

$$\int_0^3 \int_y^3 \int_0^{\sqrt{9-y^2}} dz dx dy$$

OMIT

Rewrite the integral using the order  $dz dy dx$ .

47. Find the average value of the function  $f$  over the region in the first octant bounded by the coordinate planes, and the planes  $x = 4$ ,  $y = 3$ , and  $z = 2$ .

$$f(x, y, z) = x^3 + y^2 + z^4$$

$$\frac{1}{V} \int_0^4 \int_0^3 \int_0^2 (x^3 + y^2 + z^4) dz dy dx$$

$$V = 24$$

$$\frac{1}{24} \int_0^4 \int_0^3 \left[ x^3 z + y^2 z + \frac{z^5}{5} \right]_{z=0}^{z=2} dy dx = \frac{1}{24} \int_0^4 \int_0^3 \left( 2x^3 + 2y^2 + \frac{32}{5} \right) dy dx$$

$$\frac{1}{24} \int_0^4 \left[ 2x^3 y + \frac{2y^3}{3} + \frac{32y}{5} \right]_{y=0}^{y=3} dx$$

Page 13

$$\frac{1}{24} \int_0^4 \left( 6x^3 + 18 + \frac{96}{5} \right) dx = \frac{1}{24} \left[ \frac{6x^4}{4} + \frac{186x}{5} \right]_0^4$$

$$= \frac{1}{24} \left( 6 \cdot 64 + \frac{186 \cdot 4}{5} \right) = 22.2$$

48. Evaluate the following iterated integral.

$$\int_0^{\pi/2} \int_0^{1-\cos^4 \theta} \int_0^{1-r^2} r \sin \theta \, dz \, dr \, d\theta = \int_0^{\pi/2} \int_0^{1-\cos^4 \theta} (1-r^2) r \sin \theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \sin \theta \left[ \frac{r^2}{2} - \frac{r^4}{4} \right]_{r=0}^{r=1-\cos^4 \theta} d\theta = -\frac{1}{4} \int_0^{\pi/2} (\cos^{16} \theta - 2 \cos^8 \theta) \sin \theta \, d\theta = \frac{1}{4} \left[ \frac{2u^9}{9} - \frac{u^{17}}{17} \right]_1^0$$

$$= \frac{1}{4} \left( \frac{1}{17} - \frac{2}{9} \right) = \frac{7}{102}$$

$u = \cos \theta$   
 $du = -\sin \theta \, d\theta$

49. Evaluate the following iterated integral.

$$\int_0^{\pi/3} \int_0^{\pi/3} \int_0^{\cos \theta} \rho^2 \sin \phi \cos \phi \, d\rho \, d\theta \, d\phi = \int_0^{\pi/3} \int_0^{\pi/3} \left[ \frac{\rho^3}{3} \right]_0^{\cos \theta} \sin \phi \cos \phi \, d\theta \, d\phi$$

$$\int_0^{\pi/3} \int_0^{\pi/3} \cos^3 \theta \sin \phi \cos \phi \, d\phi \, d\theta = \frac{1}{3} \int_0^{\pi/3} \left[ \frac{\cos^2 \phi}{2} \right]_0^{\pi/3} \cos^3 \theta \, d\theta = \frac{(\cos^2 \frac{\pi}{3} - 1)}{6} \int_0^{\pi/3} \cos^3 \theta \, d\theta$$

50. Evaluate the following iterated integral.

$$\int_0^{2\pi} \int_0^{\pi} \int_3^5 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \sin \phi \left[ \frac{\rho^3}{3} \right]_3^5 \, d\phi \, d\theta = \left( \frac{125}{3} - 9 \right) \int_0^{2\pi} \int_0^{\pi} \sin \phi \, d\phi \, d\theta$$

$$= 2 \left( \frac{125}{3} - 9 \right) 2\pi = \frac{392}{3} \pi$$

51. Convert the integral below from rectangular coordinates to ~~both~~ cylindrical ~~and~~ spherical coordinates, and evaluate the simpler iterated integral.

$$\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x^2+y^2}^9 x \, dz \, dy \, dx$$

cylindrical

$$\int_0^{2\pi} \int_0^3 \int_{r^2}^9 r \cos \theta \, dz \, dr \, d\theta$$

52. Convert the integral below from rectangular coordinates to ~~both cylindrical and spherical coordinates, and evaluate the simpler iterated integral.~~

$$\int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} \sqrt{x^2+y^2+z^2} \, dz \, dy \, dx = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^5 \rho^3 \sin \phi \, d\rho \, d\theta \, d\phi$$

53. Use cylindrical coordinates to find the volume inside the sphere

$$x^2 + y^2 + z^2 = 4$$

$$\int_0^{\sqrt{2}} \int_0^{2\pi} \int_r^{\sqrt{4-r^2}} r \, dz \, d\theta \, dr$$

and above the upper nappe of the cone  $z^2 = (x^2 + y^2)$ .

$$\int_0^{2\pi} \int_0^{\sqrt{2}} r(\sqrt{4-r^2} - r) \, dr \, d\theta = 2\pi \left( \left[ -\frac{1}{2} \int_{r=0}^{r=\sqrt{2}} \sqrt{u} \, du - \left[ \frac{r^3}{3} \right]_0^{\sqrt{2}} \right) = 2\pi \left( \left[ \frac{2}{3} \right]_4^{\frac{2\sqrt{2}}{3}} - \frac{2\sqrt{2}}{3} \right)$$

etc

54. Use spherical coordinates to find the volume of the solid inside the torus given by  $\rho = 9 \sin \phi$ .

OMIT

55. Use spherical coordinates to find the  $z$  coordinates of the center of mass of the solid lying between two concentric hemispheres of radii 6 and 8, and having uniform density  $k$ .

OMIT